Week 1:

Goal of this course: Learn the technique of solving system of linear equation.

trick from high school:

M1: substitution v.s M2: elimination.

M1: substitution

eg: $\begin{cases} 2x+3y = 4 & ① \\ x+y=0 & ② \end{cases}$

Put $x=-y$ into ① yields

$2(-y)+3y = 4$

$\Rightarrow y=4$ substitution again

$\Rightarrow x=-y=-4$ #

M2: elimination

$\begin{cases} 2x+3y = 4 & ① \\ x+y=0 & ② \end{cases}$

② ⟷ ①

$\Rightarrow \begin{cases} x+y=0 \\ 2x+3y=4 \end{cases}$

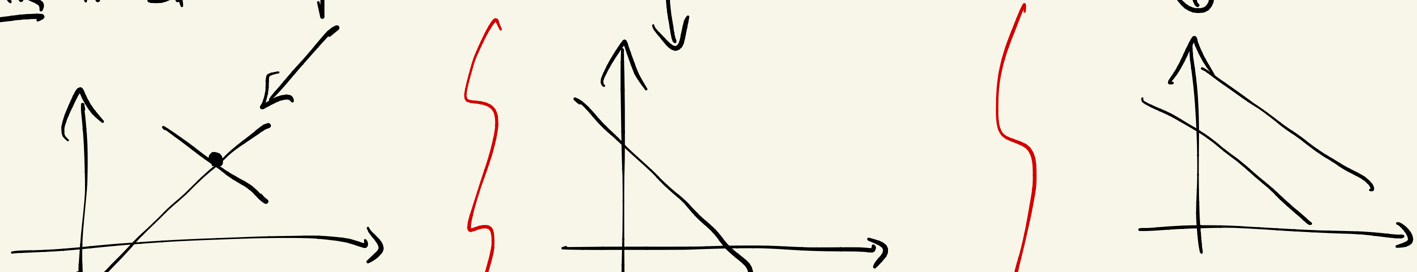
$\Rightarrow$ ② → ② − 2①

$\begin{cases} x+y=0 \\ y=4. \end{cases}$

$\Rightarrow x=-y=-4$ #.

Goal: generalize one of them to cases w/ more variables.

Rmk in 2D: possibilities



unique solution ; ∞ many solution ; No solution eg: $\begin{cases} x+y=1 \\ x-y=0 \end{cases}$ $\begin{cases} x+y=1 \\ 2x+2y=2 \end{cases}$ $\begin{cases} x+y=1 \\ x+y=2 \end{cases}$

Example of 3 variables x, y, z:

(Idea: Reduced back to 2 variables.)

Solve: $S: \begin{cases} x+2y+2z=4 & (A) \\ x+3y+3z=5 & (B) \\ 2x+6y+5z=6 & (C) \end{cases}$

Step 1: search for possible solutions

Step 2: checking

Step 3: draw conclusion

$\Rightarrow$ By substitution / elimination:

Substitution Method (?)

By (A), $x = 4-2y-2z$. Put this into (B) and (C)

(B): $(4-2y-2z)+3y+3z=5$

$\Rightarrow y+z=1$

(C): $2(4-2y-2z)+6y+5z=6$

$$\Rightarrow \quad 2y + z = -2$$

$$\begin{cases} y + z = 1 \\ 2y + z = -2 \end{cases}$$ = system of 2 linear equations w/ 2 unknown

$$\Rightarrow \begin{cases} y = -3 \\ z = 4 \end{cases}$$ By Method in 2-variables.

$$\xRightarrow{\text{substitution}} \begin{cases} x = 2 \\ y = -3 \\ z = 4. \end{cases} \quad \text{\#}$$

"Alert : Not so user friendly".

Method of elimination

$$S_0: \begin{cases} x + 2y + 2z = 4 & (A) \\ x + 3y + 3z = 5 & (B) \\ 2x + 6y + 5z = 6 & (C) \end{cases}$$

"New A, B, C".

$$\xrightarrow[C \to C - 2A]{B \to B - A} S_1: \begin{cases} x + 2y + 2z = 4 & (A) \\ y + z = 1 & (B) \\ 2y + z = -2 & (C) \end{cases}$$

$$\xrightarrow{C \to C - 2B} S_2: \begin{cases} x + 2y + 2z = 4 \\ y + z = 1 \\ \boxed{z = 4.} \end{cases}$$

substitution to find x, y !!

$$\Rightarrow \begin{cases} x = 2 \\ y = -3 \\ z = 4 \end{cases}$$

Rmk: seems easier !!

$\therefore$ Step 1: possible solution $= (2, -3, 4)$.

Step 2:
$$\begin{cases} (2) + 2(-3) + 2(4) = 4 \\ (2) + 3(-3) + 3(4) = 5 \\ 2(2) + 6(-3) + 5(4) = 6 \end{cases}$$

Step 3: Set of sol. $= \{(2, -3, 4)\}$ #.

Example w/o solutions:

Solve: S_1:
$$\begin{cases} x - 5y + 3z = 1 & A \\ 2x - 4y + z = 0 & B \\ x + y - 2z = -2 & C \end{cases}$$

(Method of eliminations)

$$\xrightarrow[C \to C - A]{B \to B - 2A} S_2: \begin{cases} x - 5y + 3z = 1 \\ 6y - 5z = -2 \\ 6y - 5z = -3 \end{cases}$$

$$\xrightarrow{C \to C - B} S_3: \begin{cases} x - 5y + 3z = 1 \\ 6y - 5z = -2 \\ \boxed{0 = -1.} \end{cases}$$

Impossible.

Step 1 : No possible solution #

Example of ∞ many solutions :

$$S_1 : \begin{cases} x + 3y + z = 4 \\ x - 2y + 2z = -9 \\ 2x + y + 3z = -5 \end{cases}$$

$$\xrightarrow[C \to 2A - C]{B \to A - B} \quad S_2 : \begin{cases} x + 3y + z = 4 \\ 5y - z = 13 \\ -5y + z = -13 \end{cases}$$

$$\xrightarrow{C \to C + B} \quad S_3 : \begin{cases} x + 3y + z = 4 \\ 5y - z = 13 \\ 0 = 0 \end{cases}$$

If $y = t$ for some $t \in \mathbb{R}$,

then By B, $z = 5t - 13$

By A and above,

$$x = 4 - (5t - 13) - 3t$$

$$= 17 - 8t.$$

Step 1 : possible solutions are

$$\{ (17-8t, t, 5t-13) \mid t \in \mathbb{R} \}$$

Step 2 :
$$\begin{cases} (17-8t) + 3t + (5t-13) = 4, \\ (17-8t) - 2t + 2(5t-13) = -9 \\ 2(17-8t) + t + 3(5t-13) = -5 \end{cases}$$ holds

for any $t \in \mathbb{R}$.

Step 3 : Solution set $= \{ (\overset{x}{\overset{\shortparallel}{17-8t}}, \overset{y}{\overset{\shortparallel}{t}}, \overset{z}{\overset{\shortparallel}{5t-13}}) \mid t \in \mathbb{R} \}$

Step 1 : Relies on elimination (step (i))

Inverted ▲

$$\begin{cases} x + 3y + z = 4 \\ x - 2y + 2z = -9 \\ 2x + y + 3z = -5 \end{cases} \longrightarrow \begin{cases} x + 3y + z = 4 \\ \quad 5y - z = 13 \\ \qquad\quad 0 = 0 \end{cases}$$

Step (ii) : Substitution to find z, x in term of y in this example.

Generalize the Idea to More Variables:

Example:

$$S_1: \begin{cases} x_2 + x_3 + 2x_4 + 2x_5 = 2 & A \\ x_1 + 2x_2 + 3x_3 + 2x_4 + 3x_5 = 4 & B \\ -2x_1 - x_2 - 3x_3 + 3x_4 + x_5 = 3 & C \end{cases}$$

Step I: find possible solutions by Elimination.

$$\xrightarrow[C \to C+2B]{A \leftrightarrow B} S_2: \begin{cases} x_1 + x_2 + 3x_3 + 2x_4 + 3x_5 = 4 \\ x_2 + x_3 + 2x_4 + 2x_5 = 2 \\ 3x_2 + 3x_3 + 7x_4 + 7x_5 = 11 \end{cases}$$

Inverted Δ

$$\xrightarrow{C \to C-3B} S_3: \begin{cases} x_1 + x_2 + 3x_3 + \boxed{2x_4 + 3x_5} = 4 \\ x_2 + x_3 + \boxed{2x_4 + 2x_5} = 2 \\ \boxed{x_4 + x_5 = 5} \end{cases}$$

x_4 in term of x_5 (great!!)

(Not stop yet)

Want to simplify ☆☆ part.

$$\xrightarrow[B \to B-2C]{A \to A-2C} S_4: \begin{cases} x_1 + 2x_2 + 3x_3 + 0 + x_5 = -6 \\ x_2 + x_3 + 0 + 0 = -8 \\ x_4 + x_5 = 5 \end{cases}$$

$A \to A - 2B$

$\longrightarrow \quad S_5: \begin{cases} \boxed{x_1} + \boxed{0} + x_3 + \boxed{0} + x_5 = 10 \\ \qquad \boxed{x_2} + x_3 + 0 + 0 = -8 \\ \qquad\qquad \boxed{x_4} + x_5 = 5 \end{cases}$

Method: Gaussian elimination.

$\therefore$ · if $x_5 = t$ for some $t \in \mathbb{R}$.

then $x_4 = 5 - t$. By $\boxed{C}$

· if $x_3 = s$ for some $s \in \mathbb{R}$.

then $x_2 = -8 - s$. By $\boxed{B}$.

· then $x_1 = 10 - t - s$. By (A)

Step 1: possible solutions

$$= \{(10 - t - s, -8 - s, s, 5 - t, t) \mid s, t \in \mathbb{R}\} \#$$

Step 2: Checking, ...

Step 3: solution sets

$$= \{(10 - t - s, -8 - s, s, 5 - t, t) \mid s, t \in \mathbb{R}\} \#$$

In General:

Defn: The system of m linear equations w/ n unknown is given by

$$S: \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + \cdots \cdots + a_{mn}x_n = b_m \end{cases}$$

Where
- $\{a_{ij}\}$ for $i = 1, 2, \cdots m$ and $j = 1, 2, \cdots, n$ are given (fixed) real numbers
- $\{b_k\}$ for $k = 1, 2, \cdots m$ are given (fixed) real numbers.

- Let $(t_1, \cdots, t_n)$ be n real number. We say that $(x_1, \cdots, x_n) = (t_1, \cdots, t_n)$ is a solution to S if

$$\begin{cases} a_{11}t_1 + \cdots + a_{1n}t_n = b_1 \\ \vdots \\ a_{m1}t_1 + \cdots + a_{mn}t_n = b_m \end{cases}$$ holds.

- S is said to be consistent if $\exists$ solution to S
  (Eg: unique sol. or ∞ many solution)
- S is inconsistent if there is No solution.

Rmk: It highly depends on $\{b_i\}$ also!!

Thm Given a system of m linear equation with n unknowns, either one of the following holds

In 2D

1. $\exists$ unique solution to S
2. $\exists$ ∞ many solution to S
3. there is No solution

Defn: Equation operation

Type (a): "(A) $\leftrightarrow$ (B)" Swapping

Type (b): "(A) $\rightarrow$ $k \cdot$(A)" Scalar multiplication.

Type (c): "(A) $\rightarrow$ (A) + k(B)" adding scalar multiple of other eqn.

Defn: Given two system of m linear equation w/ n unknowns, (S) and (T).

we say that (S) and (T) are equivalent if (S) and (T) have the same set of solutions

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Thm (~~the~~ justify what we use)

If (S) is obtained from (T) by applying finitely many equation operations of type (a), (b), or (c), then (S) and (T) are equivalent.

Example: Solve

$$S_1: \begin{cases} x_2 - 2x_3 = 1 \\ -x_1 - 2x_2 + 3x_3 = -4 \\ 2x_1 + 7x_2 - 12x_3 = 11 \end{cases}$$

$$\xrightarrow{A \leftrightarrow B} S_2: \begin{cases} -x_1 - 2x_2 + 3x_3 = -4 \\ x_2 - 2x_3 = 1 \\ 2x_1 + 7x_2 - 12x_3 = 11 \end{cases}$$

$$\xrightarrow{A \to -A} S_3: \begin{cases} x_1 + 2x_2 - 3x_3 = 4 \\ x_2 - 2x_3 = 1 \\ 2x_1 + 7x_2 - 12x_3 = 11 \end{cases}$$

$$C \to C-2A \quad \longrightarrow \quad S_4: \begin{cases} x_1 + 2x_2 - 3x_3 = 4 \\ x_2 - 2x_3 = 1 \\ 3x_2 - 6x_3 = 3 \end{cases}$$

$$C \to \tfrac{1}{3}C \quad \longrightarrow \quad S_5: \begin{cases} x_1 + 2x_2 - 3x_3 = 4 \\ x_2 - 2x_3 = 1 \\ x_2 - 2x_3 = 1 \end{cases}$$

$$C \to C-B \quad \longrightarrow \quad S_6: \begin{cases} x_1 + 2x_2 - 3x_3 = 4 \\ x_2 - 2x_3 = 1 \\ 0 = 0 \end{cases}$$

$$A \to A-2B \quad \longrightarrow \quad S_7: \begin{cases} \boxed{x_1} + 0 + x_3 = 2 \\ \boxed{x_2} - 2x_3 = 1 \\ 0 = 0 \end{cases}$$

$\boxed{\text{Thm} \Rightarrow S_i \text{ are all equivalent for } i = 1, 2, \cdots, 7.}$

Solution set for S_7 = solution set for S_1 (original)

$$\| $$

$$\{ (2-t,\ 1+2t,\ t) \mid t \in \mathbb{R} \}$$

$\#$